BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

BY

RÉMI VAILLANCOURT

ABSTRACT

This lecture gives an inside look into the proof of the continuity of pseudo-differential operators of order m and type ρ, δ_1, δ_2 for $0 \leqq \rho \leqq \delta_1 = 1$, $0 \leqq \rho \leqq \delta_2 < 1$, and $m/n \leqq \rho \leqq (\delta_1+\delta_2)/2$. Applications are mentioned.

Recently Calderón and Vaillancourt [2] established the following result: Pseudo-differential operators of order m and type ρ, δ_1, δ_2 are bounded in $L^2(R^n)$ provided that $0 \leqq \rho \leqq \delta_j < 1$, and $m/n \leqq \rho - (\delta_1 + \delta_2)/2$. In fact it is enough to require that the symbol $p(x_1, x_2, \xi)$ of the operator

$$(Pu)(x_1) = (2\pi)^{-n} \int\int e^{i(x_1 - x_2) \cdot \xi} p(x_1, x_2, \xi) u(x_2)\, dx_2\, d\xi$$

satisfy the inequalities

$$\left| \partial_{x_j}^{\alpha} \partial_{\xi}^{\beta} p(x_1, x_2, \xi) \right| \leqq c(1 + |\xi|)^{m + \delta_j|\alpha| - \rho|\beta|}, \; j = 1, 2,$$

for $0 \leqq |\beta| \leqq 2[n/2] + 2$, and $0 \leqq |\alpha| \leqq 2m_j$, where m_j is the least integer such that

$$m_j(1 - \delta_j) \geqq 5\, n/4.$$

In 1971, Hörmander [5] established that if $\rho > 0$, $m/n < \rho - (\delta_1 + \delta_2)/2$, and p has compact support in (x_1, x_2) then the pseudo-differential operator with symbol p is bounded in L^2. He also proved that this need not be true if $m/n > \rho - (\delta_1 + \delta_2)/2$. The result also fails if $\delta_1 = \delta_2 = \rho = 1$ and $m = 0$, as was shown by Chin-Hung Ching [3]. Thus [2] settles the borderline case and removes the restriction on the support of p.

In this paper I intend to give some intuitive reasons why the proof of [2] works. I feel that this is interesting and might be useful for further research.

The proof of the theorem rests on an "almost orthogonal" splitting of the operator P to which one applies a generalization of a lemma originally due to Cotlar (see [6]). I state this auxiliary lemma as given in [2].

Let A denote a bounded operator on a separable Hilbert space H and let $A(z)$ be a weakly measurable, uniformly bounded, operator-valued function on a measure space Z with measure dz. If

$$\| A^*(z_1)A(z_2) \| \leqq h_1(z_1, z_2)^2$$

$$\| A(z_1)A^*(z_2) \| \leqq h_2(z_1, z_2)^2$$

and

$$\int h_1(z_1, z)h_2(z, z_2)dz$$

is the kernel of a bounded operator on $L^2(Z)$ with norm N^2, then

$$\left\| \int_E A(z)dz \right\| \leqq N,$$

where E is any subset of finite measure of Z.

To prove the lemma one shows that, for any set E of finite measure,

$$\lim_{n\to\infty} \left\| \left[\left(\int_E A(z)dz \right) \left(\int_E A(z)dz \right)^* \right]^n \right\|^{1/n} \leqq N^2,$$

and hence

$$\| A \|^2 = \| AA^* \| = | \sigma(AA^*) | = \lim_{n\to\infty} \| (AA^*)^n \|^{1/n} \leqq N^2.$$

The insight I want to give will be achieved by replacing the continuous partition of P used in the proof of [1] and [2] by a discrete partition.

Let us consider a simple symbol of class $S^0_{0,0}$, that is, a bounded function $p(x, \xi)$ with bounded derivatives, and, for simplicity, we restrict ourselves to the one-dimensional case, $n = 1$.

Let $q(\xi)$ be a smooth function with bounded support,

$$q(\xi) \begin{cases} > 0 & \text{for } |\xi| \leqq 2/3 \\ = 0 & \text{for } |\xi| > 3/4. \end{cases}$$

Set

$$q_n(\xi) = q(\xi - n)$$

and with

$$w_n(\xi) = q_n(\xi)/[\, \Sigma\, q_n(\xi)],$$

define

$$p_n(x,\xi) = w_n(\xi)p(x,\xi).$$

Then

$$p = \sum p_n$$

and

$$p_n p_m = 0, \qquad |n-m| > 1.$$

First, the P_n are uniformly bounded,

$$\| P_n \| \leqq c,$$

since $p_n(x,\xi) \in C^2$ in ξ and has support in ξ of uniform length for all n.

Second

$$P_{2n}P^*_{2m} = 0, \qquad m \neq n,$$

since

$$p_{2n}(x,\xi)p_{2m}(x,\xi) = 0 \qquad \text{for } m \neq n.$$

Third, for $n \neq m$,

$$(v, P^*_{2n}P_{2m}u)$$

$$= (2\pi)^{-2} \int \cdots \int \bar{v}(z)e^{-i(x-z)\xi}\bar{p}_{2n}(x,\zeta)e^{i(x-y)\xi}\, p_{2m}(x,\xi)u(y)dyd\xi dxd\zeta dz$$

$$= (2\pi)^{-2} \int \cdots \int \frac{\bar{v}(z)dz}{1+(x-z)^2} e^{-i(x-z)\zeta} \frac{u(y)dy}{1+(x-y)^2} e^{i(x-y)\xi} \frac{1}{[i(\zeta-\xi)]^3}$$

$$(1-\partial_\zeta^2)(1-\partial_\xi^2)\partial_x^3[p_{2n}(x,\zeta)p_{2m}(x,\xi)]d\xi d\zeta dx.$$

Hence

$$|(v, P^*_{2n}P_{2m}u)| \leqq c\|u\| \cdot \|v\| \frac{1}{|n-m|^3}.$$

Now by the lemma,

$$\| \sum P_{2m} \| \leqq c,$$

and similarly,

$$\| \sum P_{2m+1} \| \leqq c;$$

but this implies

$$\| P \| \leqq \| \sum P_{2m} \| + \| \sum P_{2m+1} \| \leqq c < \infty.$$

This establishes the theorem when $\rho = \delta = 0$.

The above unpublished proof, due to Ching, requires only two derivatives in ξ, while three in x.

When $\rho = \delta \neq 0$, this proof requires some modifications since

$$\left| \partial_x p_{2n}(x,\zeta) p_{2m}(x,\xi) \right|$$

need not be bounded.

We first note that the pseudo-differential operators P and Q associated respectively with the symbols $p(x,\xi)$ and $p(a^{-1}x, a\xi)$, $a > 0$, have equal norm:

$$\| P \| = \| Q \|.$$

Thus for $p_n(x,\xi)$ with ξ-support around n, we choose σ such that

$$p_n(|n|^{-\sigma}x,\ |n|^{\sigma}\xi)$$

be of class $S^0_{0.0}$, namely,

$$\sigma = \delta / (1 - \delta).$$

Next we determine the proper size of the support of p_n. To do so we use the function $q(\xi)$ introduced above to define

$$q_0(\xi) = q(\xi),$$

$$q_n(\xi) = q\left(\frac{\xi - n|n|^{\sigma}}{(\sigma + 1)|n|^{\sigma}}\right), \quad |n| \geqq 1,$$

and, as before

$$w_n(\xi) = q_n(\xi) / [\Sigma q_n(\xi)],$$

$$p_n(x,\xi) = w_n(\xi) p(x,\xi).$$

Now one sees that $p_n(|n|^{-\sigma}x, |n|^{\sigma}\xi)$ is C^2 in ξ and has ξ-support of uniform length for all n; thus

$$\| P_n \| \leqq c, \text{ all } n.$$

Next, for $|n|, |m| \geqq M(\sigma)$,

$$\| P_{2n} P^*_{2m} \| = 0, \qquad n \neq m.$$

Finally, for $|n|, |m| \geqq M(\sigma)$, one shows that

$$\| P^*_{2n} P_{2m} \| \leqq \frac{c}{1 + |n - m|^{2+\varepsilon}}, \qquad \varepsilon > 0,$$

by considering the norm-equivalent symbol

$$\bar{p}_{2n}(\mu^{-\sigma}x, \mu^{\sigma}\zeta)p_{2m}(\mu^{-\sigma}x, \mu^{\sigma}\xi),$$

where

$$\mu = \min(|2n|, |2m|).$$

The same holds for odd indices. Thus, by the lemma,

$$\| P \| \leqq \sum_{|m| \leqq M} \| P_m \| + \| \sum_{|2m| > M} P_{2m} \| + \| \sum_{|2m+1| > M} P_{2m+1} \| < \infty .$$

We turn now to the proof of the theorem as given in [1] and [2]. In [1], $\rho = \delta = 0$ and the following continuous splitting of $p(x,\xi)$ was used:

$$p(x,\xi) = \iint q(x-s)q(\xi-t)g(s,t)dsdt,$$

where

$$g(x,\xi) = (1+\partial_x)^3(1+\partial_\xi)^3 p(x,\xi),$$

and

$$q(x) = \begin{cases} \frac{1}{2}x^2 e^{-x} & \text{for } x \geqq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Thus

$$(Pf)(x) = \int dsdtg(s,t)(P(s,t)f)(x)$$

where

$$(P(s,t)f)(x) = \frac{1}{2\pi}\int e^{ix\xi}q(x-s)q(\xi-t)\hat{f}(\xi)d\xi.$$

Since the function $g(s,t)$ is bounded, one need only prove that

$$\| \int P(s,t) f dsdt \| < C \| f \|$$

by means of the lemma.

For the case $\delta > 0$, the function $g(s,t)$ need not be bounded. Therefore in [2] a more involved splitting of P had to be used, namely,

$$(Pf)(x_1) = (2\pi)^{-n}\int g(x_1,x_2,\xi)e^{i(x_1-x_2)\cdot\xi}f(x_2)dx_2d\xi,$$

$$= (2\pi)^{-n}\int P(\xi)f(x_1)d\xi$$

where

$$g(x_1,x_2,\xi) = [1+(-\Delta_\xi)^N[1+|\xi|^2]^{N\rho}]\{p(x_1,x_2,\xi)\cdot$$
$$\cdot [1+[1+|\xi|^2]^{N\rho}|x_1-x_2|^{2N}]^{-1}\},$$

and

$$N = [n/2]+1.$$

The lemma is now applied to $P(\xi)$.

We remark that this last splitting of P holds for any dimension n. In the case of the discrete splitting of P, one can partition R^n into 2^n disjoint sets of cubes and apply the argument to each set.

In concluding, I would like to mention an important application of the theorem treated in this paper. Beals and Fefferman, at the University of Chicago, have extended the sufficient condition of Nirenberg and Treves [9] for the local solvability of differential operators of principal type with analytic leading coefficients to the case of infinitely smooth coefficients. In the proof they need the fact that a pseudo-differential operator of order 0 and type 1/2 , 1/2 be bounded.

I would finally remark that, using the method of proof outlined in this paper one can show that an operator of order 0 and of type ρ and δ, $1 \geqq \rho > \delta \geqq 0$ is bounded provided its symbol $p(x,\xi)$ satisfies the inequalities

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leqq c(1+|\xi|)^{\delta|\alpha|-\rho|\beta|}$$

for $0 \leqq |\beta| \leqq$ const. n and $0 \leqq |\alpha| \leqq$ const. $n/(1-\delta)$. This contrasts with previous proofs [4, 7 and 8] where the number of derivatives grew as $n/(\rho-\delta)$.

In case $\rho = \delta_1 = \delta_2$, one can show, by using the methods of [2] and [8] that the operators of order zero form an algebra.

REFERENCES

1. A. P. Calderón and R. Vaillancourt, *Ou the boundedness of pseudo-differential operators*, J. Math. Soc. Japan **23** (1971), 374–378.

2. A. P. Calderón and R. Vaillancourt, *A class of bounded pseudo-differential operators*, Proc. Nat. Acad. Sci. U. S. A. **69** (1972), 1185–1187.

3. C.-H. Ching, *Pseudo-differential operators with nonregular symbols*, J. Differential Equations **11** (1972), 436–447.

4. L. Hörmander, *Pseudo-differential operators and hypoelliptic equations*, Proc. Symp. Pure Math. (Amer. Math. Soc., Providence, R. I.) **10** (1967), 138–183.

5. L. Hörmander, *On the L^2 continuity of pseudo-differential operators*, Comm. Pure Appl. Math. **24** (1971), 529–535.

6. A. W. Knapp and E. M. Stein, *Singular integrals and the principal series*, Proc. Nat. Acad. Sci. U. S. A. **63** (1969), 281–284.

7. H. Kumano-go, *Remarks on pseudo-differential operators*, J. Math. Soc. Japan **21** (1969), 413–439.

8. H. Kumano-go, *Algebras of pseudo-differential operators*, J. Fac. Sci. Univ. Tokyo **17** (1970), 31–50.

9. L. Nirenberg, and F. Treves, *On local solvability of linear partial differential equations, Part II, Sufficient conditions*, Comm. Pure Appl. Math. **23** (1970), 459–509.

THE UNIVERSITY OF OTTAWA
OTTAWA, CANADA